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IS THERE A CONTINUUM AMBIGUITY FOR ELASTIC πN AMPLITUDES?(+)

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(December 1980)

Abstract

The continuum ambiguity is defined as a phase factor not determined by those amplitude zeros near the physical region that can be directly deduced from the data; such a factor may be approximated by a polynomial whose zeros are far from the physical region. A study of recent πN partial wave analysis (CUTKOSKY76 and HOHLER78) reveals that such a phase is either null or negligeable; CUTKOSKY76' s amplitude is found similar to that of a partial wave analysis based on Barrelet zeros. We give general arguments based on the notion of "peripheral resonances" to explain this situation. Our arguments imply that Atkinson"s π^+p "continuous ambiguity" is not relevant to the reliability of Barrelet-zero amplitude analysis.

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The problem of the "continuum ambiguity" has long been a thorn in the side of the partial wave analysis[1]; an angle-dependent phase factor does not manifest itself through data which depends only on the amplitude modulus. Burkhardt and Martin[2] have shown that general unitarity-analyticity S-matrix properties in principle determine a unique amplitude from the data, but a simple incorporation of unitarity-analyticity into data analysiss has never been achieved . Efforts to resolve the continuum ambiguity have led to complex and expensive analyses[3-5] that employ dispersion relations in conjunction with data fitting. Data amalgamation is required[3], mixing different experiments that may have quite different systematic errors. The error of the obtained amplitude is difficult to assess[4]. In sharp constrast, Barrelet data analysis[6,7] deduces directly from individual experiments, up to a "discrete ambiguity"[6], those amplitude zeros that lie close to the physical region. Assuming that the discrete ambiguity is resolvable, the question arises: To what extent do "nearby" (Barrelet) zeros determine the amplitude phase? The question is made precise through Barrelet 's formula for a spin0-spin 1/2+ elastic amplitude approximated by a finite number of nearby zeros[8]. Most partial wave analysts[9] believe that the Barrelet formula[6] (to be stated below) with only nearby zeros[10] does not give the phase "adequately" and that dispersion relations are needed. In this paper, we present new arguments and evidence to support our long-standing belief[10] that for π^+p elastic scattering the Barrelet formula with nearby zeros only, gives as much information about the phase as expensive data analysis assisted by dispersion relations[3,4]. The connection between data error and amplitude error is then straightforward[10c].

To begin, we emphasize that in resolving the discrete ambiguity[6-7,10] it is necessary implicitly to invoke analyticity in energy, although not through dispersion relations. Analyticity in angle is implicit in the polynomial representation of angular dependence. Unitarity is recognized[11] to some extent by phase factors correlated with zeros in the Barrelet formula as well as by positivity requirements on the imaginary parts of polynomial coefficients. We are not, therefore, proposing to ignore unitarity-analyticity.

A second and crucial point needs to be made: Elastic πN data has turned out to be almost completely representable by zeros[8] whose presence is manifested by angular structure in the data[6]. Such zeros we shall, for clarity, call "Barrelet zeros" or "nearby" zeros. The Barrelet formula successfully represents the observed modulus of the πN amplitude almost entirely in terms of nearby zeros[7,10]. "Data zeros" , a more general term, come from any polynomial approximation to the data. They may be close to the physical region (as in πN) or far away (as in K^+p). But only when nearby, can data zeros be identified with true amplitude zeros[8].

Contrast with K^+p elastic scattering is helpful to understand the remarkable nature of this point. There is no angular structure in K^+p elastic scattering and correspondingly there are no nearby (Barrelet) zeros[12]. Obviously the data can be fitted through a finite-order polynomial-representation of the K^+p amplitude, and any such representation has zeros, but such zeros are "distant" and artificial; they do not represent true amplitude zeros. A pole function with no zeros at all can also fit K^+p data. It is a remarkable and special feature of elastic πN scattering that the data can be approximated by Barrelet zeros which each correspond to observable angular structure. It is not necessary to include distant zeros or pole-type angular dependence in order to represent the πN elastic amplitude modulus, at least up to 2.5 GeV.

We roughly understand why such should be the case through the prominence of "peripheral" πN resonances. (There are no resonances in elastic K^+p scattering). In fact, πN resonances are observed for all angular momenta up to the maximum J that is significant at any given energy. At least as far as amplitude modulus is concerned, the partial wave expansion cuts off rapidly after some resonating J_{max} whose contribution to the data is prominent. The data (up to 2.5 GeV) require no long high-J tail of nonresonating πN partial waves[7,10], so the zeros tend to be near those of $R_{Jmax,\varepsilon}$ -- the pseudopolynomial (see below) belonging to the dominant resonance. These zeros are close to the physical region.

The continuum ambiguity has been formulated by Burkhardt[1] as a phase factor $e^{i\phi}$, where ϕ is an analytic angular function real in the physical region, that can multiply any amplitude whose modulus fits the data. For our purposes, we start with the Barrelet formula with "nearby zeros" only and consider the possibility of such an extra phase. Such a factor can be given a polynomial approximation, with associated artificial zeros that are "distant" from the physical region (by definition, they do not affect the modulus!) Multiplying such a polynomial factor into the Barrelet formula, with a finite set of N nearby zeros corresponding to $J_{max} = (N+1)/2$ (see below), adds higher partial waves, while at the same time changing the partial waves for $J < J_{max}$. The nearby zero positions, of course, are not shifted since they are determined by the amplitude modulus. But a common-sense question is : Why should a tail of high partial waves not affect the modulus as well as the phase? Or equivalently, is not uniform smallness of partial waves for $J > J_{max}$ the most reasonable explanation for the representability of the πN modulus by nearby zeros? We shall give detailed arguments based on unitarity-analyticity that make any alternative unlikely.

Direct and persuasive evidence is that "standard" πN partial wave analyses[3,4] using dispersion relations, have produced amplitudes with a phase identical to that obtained from using the Barrelet amplitude formula with only nearby zeros[13]. There is no significant contribution in one of these recent analyses[3] from a $J > J_{max}$ tail of high partial waves; for that analysis, as in Ref. [10], the nearby zeros control both the modulus and the phase of the amplitude.

A nagging question remains about the work of Atkinson[14], who claimed to establish the importance of the "continuum ambiguity" in $\pi^{+}p$ scattering. Atkinson did not introduce the continuum ambiguity through a multiplicative phase factor, however, and it is possible that by allowing a small change in the amplitude modulus, he was led to confuse the "standard" (Burkhardt) continuum ambiguity[1] discussed here with the discrete ambiguity. We also suspect that Atkinson allowed excessively large imaginary parts for peripheral (large ℓ) partial waves (see below). The situation is further confused by the fact that Atkinson's ambiguity is significant only in very-low J partial waves that make a small contribution to the amplitude modulus. Such waves are, at best, inaccurately determined by the data.

These considerations support our contention that the Barrelet amplitude formula dominated by "nearby zeros" , provides a πN amplitude that is as reliable (sometimes more reliable) as some generated by more complex analyses with dispersion relations, and that carries experimental errors straightforward to evaluate[10c].

Mathematical formulation of the continuum ambiguity is simple and unambigous in the Barrelet formalism[1,9]. The Barrelet amplitude formula[6] is a mathematical identity[13] which expresses any polynomially-approximated amplitude (i.e. a finite number of partial waves) through its N zeros in the variable $w=e^{i\theta}$:

$$F(s,w) \simeq F(s,w_o) \frac{w_o^{\,p}}{w^{\,p}} \prod_{j=1}^{N} \frac{w - w_j}{w_o - w_j} \qquad (1)$$

where w_o, the point of normalization , is usually taken to be the forward direction. The power p is equal to N/2 (if N is even) or (N+1)/2 (if N is odd). As explained later, it is physically appropriate to consider only the case of even N. A value for p different from N/2 has been shown to violate the positivity of partial-wave imaginary parts[11]. The zeros can be found either from straightforward moment analysis of the data supplemented by requirements of analyticity in energy to resolve the discrete ambiguity[7,10] or from a polynomial fit to the data with dispersion-relation constraints[3-5]. Whatever their origin, the zeros satisfy the data constraint:

$$\Sigma(s,w) = \frac{d\sigma}{d\Omega}\ (1 + P) = F(s,w)\ \ \overline{F(s,\ \overline{w^{-1}}}\ \ \simeq \Sigma(s,w_o)\ \frac{1}{w^N} \prod^{N} \frac{(w - w_i)(w - \overline{w_i^{-1}})}{(w_o - w_i)(w_o - \overline{w_i^{-1}})} \qquad (2)$$

where w_i or its mirror-image $\overline{w_i^{-1}}$ represents w_j in Eq. (1).

From πN moment analysis "data zeros" turn out to be "nearby"[8]. In the case of elastic K^+p (studied in detail up to 2.0 GeV by Urban[12], also employing the Barrelet moment method of data analysis[6]), the data may still be represented by a polynomial. However the "data zeros" are unstable and can all be located outside the domain defined by the first angular singularity; such a configuration of zeros is equivalent to fitting the data with zeroless functions[12] .

πN partial wave analyses that use dispersion relations[3-5] generally produce more amplitude zeros than are required through Formula (2) by the measured data, whose statistical precision controls the number of "data zeros". In such analyses, not only nearby stable (Barrelet) zeros but also distant artificial zeros are generated, corresponding to the inclusion of partial waves with $J > J_{max}$. The issue addressed in this paper is whether inclusion of such distant zeros in πN analysis has significant impact on the (resonating) partial waves for $J \leq J_{max}$.

To begin, let us connect the Barrelet amplitude formula (1) to the amplitude partial wave expansion. Suppose that at some energy the exact set of partial waves $T_{J,\varepsilon}$ ($\ell = J-1/2$ for $\varepsilon=+$, $\ell=J+1/2$ for $\varepsilon=-$) is known but the amplitude is approximated by truncating the (exponentially-convergent) partial wave expansion at the lowest ℓ_{max} sufficiently large that in the physical region the residual terms contribute less than the experimental error (say 10%) (ℓ_{max} corresponds to a classical radius R by $\ell_{max} \simeq kR$) :

$$F^{\ell_{max}}(w) = \sum_{J,\varepsilon}^{J_{max}=\ell_{max}+1/2} T_{J,\varepsilon} \cdot R_{J,\varepsilon}(w) \tag{3}$$

This approximate amplitude satisfies the Barrelet formula (1) with $p=\ell_{max}$ and $N=2J_{max}-1 = 2\ell_{max}$. The set of N zero positions in Formula(1) plus the normalization factor is completely equivalent to the N+1 partial waves $T_{J\,\varepsilon}$

up to $\ell=\ell_{max}$. What happens if the maximum ℓ (and the maximum J) in the partial wave expansion is increased by 1 ? We know from the convergence of the partial wave expansion that $F^{\ell_{max}+1}(w)$ differs from $F^{\ell_{max}}(w)$ everywhere in the physical region by less than the experimental error. How does the Barrelet formula accomodate this fact when two new zeros must appear and, at the same time, all previous zeros are shifted?

The two new zeros will be far from the physical region, one located at w_L, a large distance from the origin of the w complex plane, and one located at w_s close to the origin. The associated new factor in Barrelet formula:

$$\frac{1 - w/w_L}{1 - w_o/w_L} \; \frac{1 - w_s/w}{1 - w_s/w_o} \quad , \qquad (4)$$

is correspondingly close to unity. At the same time, each of the original $2\ell_{max}$ zeros is shifted, but <u>those zeros within the domain of convergence</u> of the partial wave expansion cannot be shifted much. In other words, the Barrelet zeros are "stable". The same general pattern repeats each time ℓ_{max} is increased by 1, the location of the Barrelet zeros depending only slightly on the cutoff in ℓ so long as $\ell > \ell max$.

Working with a "standard" π^+p amplitude, generated so as to assure exponential partial wave convergence[3a], we have verified the foregoing scenario. At 1.395 GeV/c ($\sqrt{s} \simeq 1880$ MeV). this amplitude has 6 Barrelet zeros as shown in Fig. 1(a). These zeros are almost stationary as the upper limit in the partial-wave expansion is increased from a maximum ℓ of 3 to a maximum much higher. This example also illustrates the special and remarkable feature of π^+p scattering that the number of Barrelet zeros is equal to $2\ell_{max}$. In other words, the experimental π^+p data is represented to within about 10% by (stable) Barrelet zeros. As remarked earlier, this fact reflects the presence of peripheral resonances.

Another well-known π^+p partial wave analysis[4b] does not exhibit quite such a high degree of stability of its Barrelet zeros. At approximatly the same energy as the preceding example (Plab= 1.430 GeV/c, $\sqrt{s} \simeq 1896$ MeV), one of the six zeros moves substantially as the ℓ cutoff is changed from 3 to 9. Fig. 1(c), however, which exhibits the shift in this zero's position as higher partial waves are successively added, does not accord with exponential convergence for the partial-wave expansion -- as required by analyticity in angle. The shift accompanying the $\ell=8$, $J=15/2$ partial wave, for example, is ~3 times larger in magnitude than that accompanying the $\ell=6$, $J=11/2$ wave even though these waves have the same signature and naturality. Ref. [3] has apparently been more successful that Ref. [4] in respecting the constraints of angular analyticity. The instability shown in Fig. 1(c) reflects also the fact that distant zeros are substantially affecting the amplitude modulus. Fig. 2 (a and b) compares the fit to the data to that given by the first six zeros of Ref. [3a] with that given by the first six zeros of Ref. [4b].

Several years ago, the above relationship between distant and nearby zeros was noticed[6,10a] in the course of resolving the discrete ambiguity for π^+p elastic scattering. Starting with moment analysis of data in the $\Delta(1238)$ region, where elastic unitarity leaves no ambiguity about the two Barrelet zeros ($\ell_{max} = 1$) and then gradually raising the energy, it was empirically found that the new data zeros successively appear in pairs "at a distance" as described above, and then move toward the physical region. As soon as a pair enters the domain of convergence, thereby becoming stable Barrelet zeros, it turns out that one or more new resonances is near and that the analyticity-unitarity constraints implicit in the Breit-Wigner formula resolve the discrete ambiguity. The remarkable feature of π^+p scattering, which we have here repeatedly stressed, is that at any given energy only a small proportion of zeros lie in a "no-man's land" -- sufficiently close to the physical region to measureably affect the data (through the absolute-value squared of Formula (4)) but not yet stable Barrelet zeros. The data is always dominated by N_B nearby stable Barrelet zeros connected to resonating partial waves. The "stability" of these N_B nearby zeros means that if all moments corresponding to $\ell > N_B/2$ are dropped, in effect ignoring all distant zeros and fitting the data entirely by "large" $\ell \leqslant N_B/2$ partial waves, the nearby zero positions are not much changed.

What has the Burkhardt continuum ambiguity to do with the foregoing considerations? Suppose that π^+p data at some energy has been moment analyzed to produce a set of "data zeros", N_B of which are stable Barrelet zeros corresponding to resonances occuring up to $\ell_{max} = N_B/2$, and that the discrete ambiguity has been resolved for all zeros by requiring smooth zero trajectories as a function of energy. Suppose also that each partial-wave amplitude belonging to this set of data zeros lies within the unitarity-prescribed unit circle of the Argand diagram (with positive imaginary part). Is it possible to multiply the Barrelet formula (1) belonging to this zero set by a phase factor $e^{i\Phi(w)}$, where Φ is a real analytic function in the physical region ($0 < \theta < 2\pi$) that varies substantially over the physical region. By "substantial" let us agree to mean more than about 10%.

Assuming $\Phi(w)$ to be analytic in that domain around the physical region guaranteed to be singularity free by general S-matrix principles, we can approximate $e^{i\Phi(w)}$ in this domain by a convergent pseudopolynomial of the Barrelet form (1), introducing pairs of artificial distant zeros as factors of the form (4) to be added to the original Barrelet formula. The minimum number of such pairs needed to approximate the phase factor (without changing the modulus) will depend on the magnitude of the phase variation over the physical region. This number determines how many additional important ℓ values are introduced above the original $\ell_{max} = N_B/2$. Since the observed values of N_B already corresponds to the ℓ_{max} expected from geometrical considerations, a large number of additional important partial waves is unreasonable. To this extent there is an immediate qualitative bound on the magnitude of the continuum ambiguity. The bound may be sharpened by considering the phase of "peripheral" ($\ell > \ell_{max}$) partial waves.

It is expected from analyticity-unitarity considerations, such as imposed in Ref.(3), that for $\ell > \ell_{max}$, partial-wave amplitudes are not only small but approximately real. The largeness of a partial-wave imaginary part is a resonance signal. Now if one introduces new large-ℓ partial waves entirely through a continuum-ambiguity phase factor, the imaginary part of these waves will not be small compared to their real part. On the other hand, if a resonance is involved the associated zeros will be Barrelet zeros and will have been revealed by the modulus of the amplitude. The only way to avoid contradictions is to have the continuum-ambiguity extra $\ell > \ell_{max}$ contributions lie <u>on top of</u> $\ell > \ell_{max}$ peripheral partial waves that are <u>already present</u> in the Barrelet amplitude formula as a result of small, but observable angular structure in the data. Assuming these peripheral partial waves to be primarily real, they may be able to tolerate a modest increase in their imaginary parts. But now, we are reduced to a perturbation of a peripheral amplitude component that is already supposed to contribute less than 10% to the modulus. The continuum-ambiguity phase factor is then representable by a single pair of distant zeros through the form (4) where $|w_L| > 10$ and $|w_s| < 1/10$, and the effect on low partial waves is insignificant.

The expected negligible effect on amplitude phase from distant zeros is confirmed by the results of both Ref. [3a] and [4b]. At an energy around the J=7/2 Δ (1900) mass resonance for example, the phase generated by the collective effect of all zeros beyond the first 6 is shown in Figs. 2(c) and 2(d) as a function of angle. It is very small in comparison with the phase given by any extra "data zeros"[13]; it is even found zero in some large angle interval as can be expected from a set of zeros paired in "mirror conjugate" positions.

The ambiguity described by Atkinson[14] is not a clear example of the continuum ambiguity discussed above, because Atkinson introduces new "data zeros" and simultaneously shifts the original zeros. Any increase in the number of zeros should respect the foregoing constraints on peripheral ($\ell > \ell_{max}$) partial waves, but it is difficult to tell whether Atkinson's "continuum ambiguity" contained an excessively large peripheral imaginary part, because the effect on resonating partial waves was found to be significant only for $\ell=0$ at an energy where the $\ell=0$ partial wave is a small part of the full amplitude. We contend that if any "continuum ambiguity" is sufficiently small as not to disturb unitarity-analyticity requirements on $\ell > \ell_{max}$ then it is buried within experimental uncertainty. It may be that Atkinson has inadvertently invoked a discrete ambiguity in conjunction with displacement of the original zeros as new zeros are introduced. The continuum ambiguity, by definition, does not change the original (data) zeros, but if a small change in modulus is tolerated (inevitable in Atkinson's numerical approach), then a shift of the original zeros can be compensated by new zeros .

In conclusion, we have shown that amplitude analyses [3,4] which employ dispersion relations to determine the "continuum ambiguity" phase factor yield π^+p amplitudes differing from each other by as much as they differ from a Barrelet amplitude where (by definition) the phase is determined by the zeros of the modulus (data) only [9,10]. We have argued that such a situation is inevitable whenever peripheral resonances play a dominant role. Practical consequences for π^+p analysis are : (a) amalgamation of data from different experiments of different quality can be avoided (b) Data error can be transmitted straightforwardly into amplitude error. (c) Analysis cost can be reduced.

Aknowledgement

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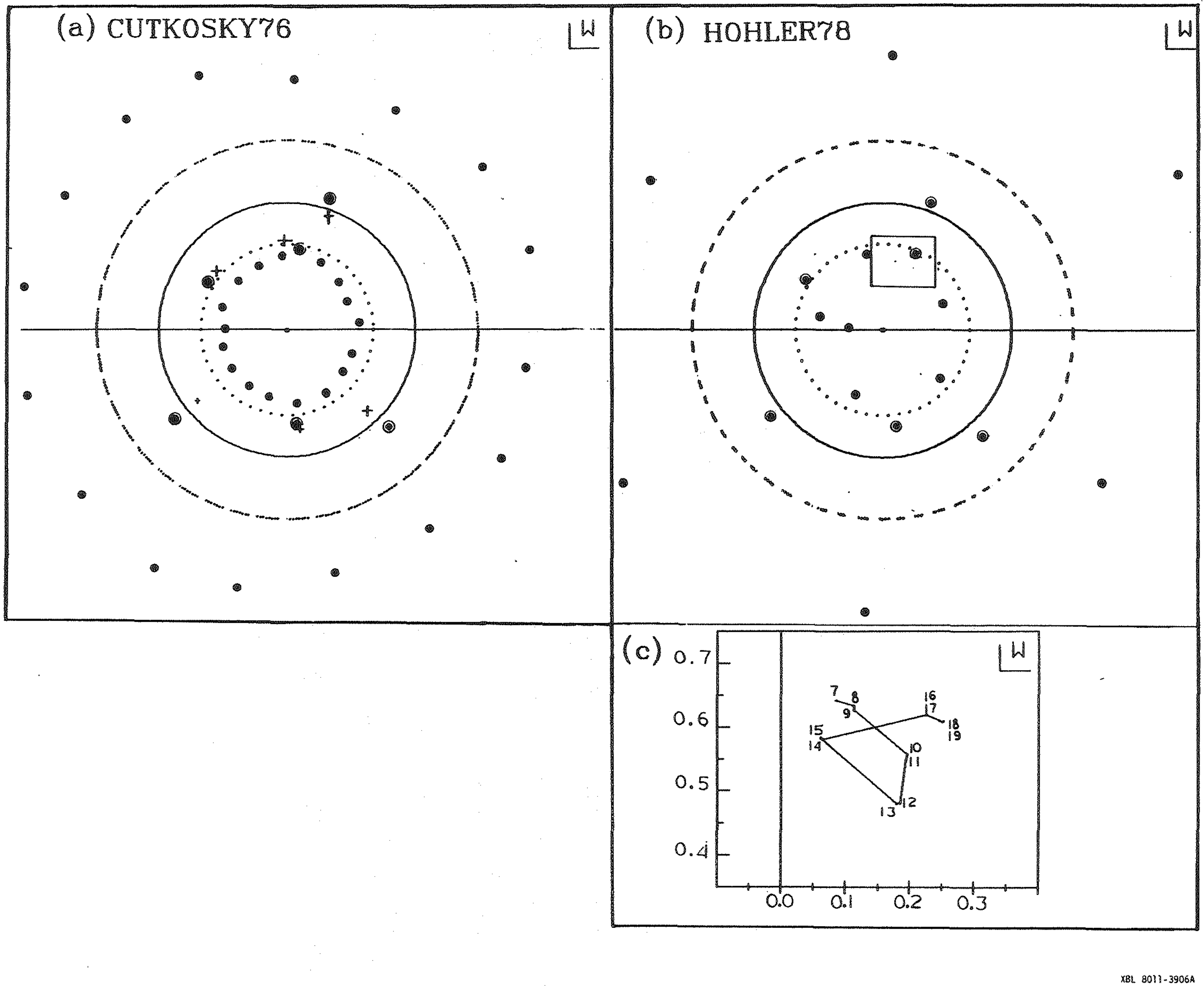
FIGURE CAPTIONS

1. Amplitude zeros in the w-plane for (a) Ref. [3a], at Plab= 1.395 GeV/c; (b) Ref. [4b], at Plab=1.430 GeV/c; (c) same as (b) but for one data-zero (located inside the rectangle of Fig. 1(b), enlarged here) as it moves as a function of the number (indicated) of partial waves kept for the amplitude.

2. Comparison of the results obtained in the analyses of Ref. [3a] and [4b] : In (a) and (b), the continous curves are the reconstructed "data" from these analyses respectively; the dotted curves are the contribution of the 6 closest zeros . The ratio of these two curves is the contribution to the modulus of the (amplitude) zeros which are located outside the crown of convergence and would therefore represent the "continuum ambiguity" . In (c) and (d) are the respective phases contributed by these latest ("non-Barrelet") zeros to the amplitude phase, as a function of the scattering angle.


(a) CUTKOSKY76
W
(b) HOHLER78
W
(c)
W
0.7
0.6
0.5
0.4
0.0
0.1
0.2
0.3
7
8
9
10
11
12
13
14
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XBL 8011-3906A

Figure 1.

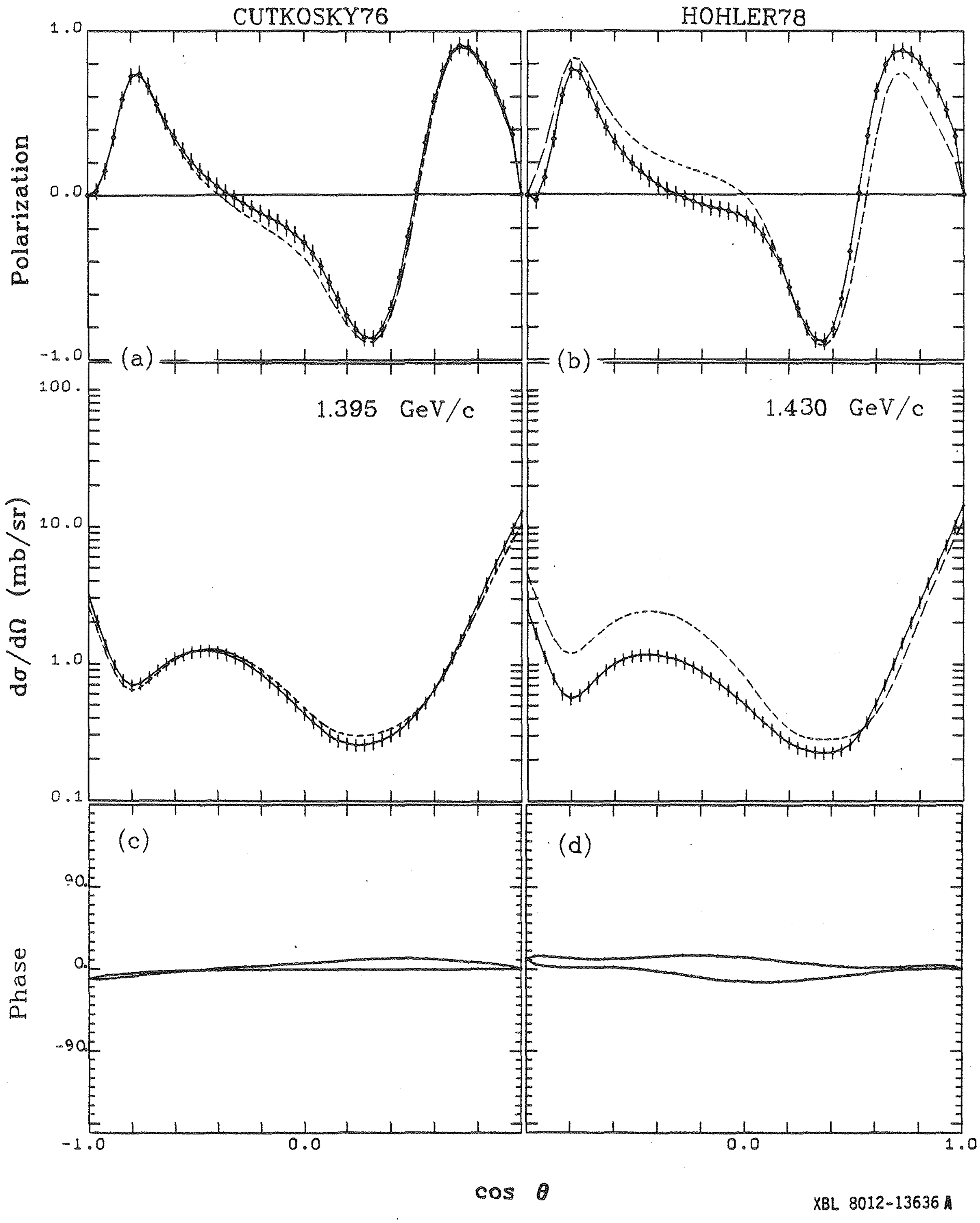
CUTKOSKY76
HOHLER78
1.0
0.0
-1.0
Polarization
(a)
(b)
100.
10.0
1.0
0.1
$d\sigma/d\Omega$ (mb/sr)
1.395 GeV/c
1.430 GeV/c
(c)
(d)
90.
0.
-90.
Phase
-1.0
0.0
1.0
cos θ
XBL 8012-13636 A

Figure 2.